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On the Lebesgue constant of Berrut's rational interpolant at equidistant nodes

Len Bos[1], Stefano De Marchi[2], Kai Hormann[3]

[1] Department of Computer Science, University of Verona, Italy
[2] Department of Pure and Applied Mathematics, University of Padua, Italy
[3] Faculty of Informatics, Università della Svizzera italiana, Switzerland

Abstract

It is well known that polynomial interpolation at equidistant nodes can give bad approximation results and that rational interpolation is a promising alternative in this setting. In this paper we confirm this observation by proving that the Lebesgue constant of Berrut's rational interpolant grows only logarithmically in the number of interpolation nodes. Moreover, the numerical results show that the Lebesgue constant behaves similarly for interpolation at Chebyshev as well as logarithmically distributed nodes.

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1 Introduction

Suppose we want to approximate a function $f\colon [a,b] \to \mathbb{R}$ by some function g that interpolates f at the $n+1$ distinct *interpolation nodes*

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Given a set of *basis functions* b_i which satisfy the *Lagrange property*

$$b_i(x_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

the interpolant g can be written as

$$g(x) = \sum_{j=0}^{n} b_j(x) f(x_j).$$

The *Lebesgue constant* of this interpolation operator is

$$\Lambda_n = \max_{a \le x \le b} \Lambda_n(x),$$

where $\Lambda_n(x)$ is the associated *Lebesgue function*

$$\Lambda_n(x) = \sum_{j=0}^{n} |b_j(x)|. \tag{1}$$

The Lebesgue constant has been studied intensively in the case of polynomial interpolation, that is, when b_i are the Lagrange basis polynomials (see [3, 4, 9] and references therein). In the special case of equidistant nodes, the Lebesgue constant for polynomial interpolation grows exponentially [8, 10, 11], which is one of the reasons why other interpolation methods should be used in this setting. One popular alternative is rational interpolation and two recent results by Carnicer [5] and Wang, Moin, and Iaccarino [12] confirm that rational interpolation at equidistant nodes can have a much smaller Lebesgue constant than polynomial interpolation. However, both papers report only numerical observations and do not give any theoretical bounds.

In this paper we investigate the rational interpolant that was introduced by Berrut [1] with basis functions

$$b_i(x) = \frac{(-1)^i}{x-x_i} \Bigg/ \sum_{j=0}^{n} \frac{(-1)^j}{x-x_j}, \qquad i = 0,\dots,n \tag{2}$$

and show that the associated Lebesgue constant grows logarithmically in the number of interpolation nodes. More precisely, we prove in Section 2 that the Lebesgue constant is bounded by $2+\ln(n)$ from above and asymptotically by $\frac{2}{\pi}\ln(n+1)$ from below, which improves the lower bound given by Berrut and Mittelmann [2]. The more interesting bound is of course the upper bound as it gives information on the stability of the interpolation process and the conditioning of the interpolation problem.

The numerical results in Section 3 further indicate that the Lebesgue constant of Berrut's rational interpolant at equidistant nodes is even smaller than the Lebesgue constant for polynomial interpolation at Chebyshev nodes. Moreover, we observe that the Lebesgue constant of Berrut's interpolant behaves similarly if Chebyshev or logarithmically distributed nodes are considered instead of equidistant nodes.

2 Main result

Let us start by recalling some well-known bounds for the partial sums of the Leibniz series and the harmonic series, namely

$$\frac{\pi}{4} - \frac{1}{2n+3} \le \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \le \frac{\pi}{4} + \frac{1}{2n+3} \tag{3}$$

and

$$\ln(n+1) \le \sum_{k=1}^{n} \frac{1}{k} \le \ln(2n+1) \tag{4}$$

for any $n \in \mathbb{N}$. Moreover, it follows from (4) that

$$\sum_{k=0}^{n} \frac{1}{2k+1} = \sum_{k=1}^{2n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{2k} \ge \ln(2n+2) - \frac{1}{2}\ln(2n+1) \ge \frac{1}{2}\ln(2n+3). \tag{5}$$

We are now ready to prove our main result, that the Lebesgue constant of Berrut's interpolant at equidistant nodes grows logarithmically in the number of nodes, by establishing logarithmic upper and lower bounds. For simplicity we assume without loss of generalization that the interpolation interval is $[0,1]$, so that the interpolation nodes are equally spaced with distance $h = 1/n$, that is,

$$x_j = jh = \frac{j}{n}, \qquad j = 0,\dots,n. \tag{6}$$

Our first result concerns the lower bound of the Lebesgue constant.

Theorem 1. *The Lebesgue constant for interpolation with the basis functions $b_i(x)$ in (2) at the nodes x_j in (6) satisfies*

$$\Lambda_n \ge c_n \ln(n+1)$$

for $c_n = 2n/(4+n\pi)$ with $\lim_{n\to\infty} c_n = 2/\pi$.

Proof. By the general definition of the Lebesgue function in (1) we have

$$\Lambda_n(x) = \frac{\displaystyle\sum_{j=0}^{n} \frac{1}{|x-j/n|}}{\left|\displaystyle\sum_{j=0}^{n} \frac{(-1)^j}{x-j/n}\right|} = \frac{\displaystyle\sum_{j=0}^{n} \frac{1}{|2nx-2j|}}{\left|\displaystyle\sum_{j=0}^{n} \frac{(-1)^j}{2nx-2j}\right|} =: \frac{N(x)}{D(x)}$$

for the basis functions in (2) and the nodes in (6). Our goal now is to bound the numerator $N(x)$ from below and the denominator $D(x)$ from above.

Let us first assume that n is even, say $n=2k$, and let $x=(n+1)/(2n)$. Using (5) we get

$$\begin{aligned}
N\left(\frac{n+1}{2n}\right) &= \sum_{j=0}^{n} \frac{1}{|n+1-2j|} = \sum_{j=0}^{2k} \frac{1}{|2(k-j)+1|} = \sum_{j=0}^{k} \frac{1}{|2(k-j)+1|} + \sum_{j=k+1}^{2k} \frac{1}{|2(k-j)+1|} \\
&= \sum_{j=0}^{k} \frac{1}{2j+1} + \sum_{j=0}^{k-1} \frac{1}{2j+1} \\
&\geq \frac{1}{2}\ln(2k+3) + \frac{1}{2}\ln(2k+1) \\
&\geq \ln(2k+1) = \ln(n+1)
\end{aligned}$$

for the numerator, and by the triangle inequality and (3) we get

$$\begin{aligned}
D\left(\frac{n+1}{2n}\right) &= \left|\sum_{j=0}^{n} \frac{(-1)^j}{n+1-2j}\right| = \left|\sum_{j=0}^{2k} \frac{(-1)^j}{2(k-j)+1}\right| = \left|\sum_{j=0}^{k} \frac{(-1)^j}{2(k-j)+1} + \sum_{j=k+1}^{2k} \frac{(-1)^j}{2(k-j)+1}\right| \\
&\leq \left|(-1)^k \sum_{j=0}^{k} \frac{(-1)^j}{2j+1}\right| + \left|(-1)^k \sum_{j=0}^{k-1} \frac{(-1)^j}{2j+1}\right| = \sum_{j=0}^{k} \frac{(-1)^j}{2j+1} + \sum_{j=0}^{k-1} \frac{(-1)^j}{2j+1} \\
&\leq \left(\frac{\pi}{4} + \frac{1}{2k+3}\right) + \left(\frac{\pi}{4} + \frac{1}{2k+1}\right) \\
&\leq \frac{\pi}{2} + \frac{2}{2k+1} = \frac{\pi}{2} + \frac{2}{n+1},
\end{aligned}$$

for the denominator. Therefore,

$$\Lambda_n\left(\frac{n+1}{2n}\right) = \frac{N(\frac{n+1}{2n})}{D(\frac{n+1}{2n})} \geq \frac{2\ln(n+1)}{\pi + \frac{4}{n+1}}. \tag{7}$$

Similarly, if n is odd, say $n=2k+1$, then at $x=1/2$ we have

$$\begin{aligned}
N\left(\frac{1}{2}\right) &= \sum_{j=0}^{n} \frac{1}{|n-2j|} = \sum_{j=0}^{2k+1} \frac{1}{|2(k-j)+1|} = \sum_{j=0}^{k} \frac{1}{|2(k-j)+1|} + \sum_{j=k+1}^{2k+1} \frac{1}{|2(k-j)+1|} = 2\sum_{j=0}^{k} \frac{1}{2j+1} \\
&\geq \ln(2k+3) = \ln(n+2)
\end{aligned}$$

and

$$\begin{aligned}
D\left(\frac{1}{2}\right) &= \left|\sum_{j=0}^{n} \frac{(-1)^j}{n-2j}\right| = \left|\sum_{j=0}^{2k+1} \frac{(-1)^j}{2(k-j)+1}\right| = \left|\sum_{j=0}^{k} \frac{(-1)^j}{2(k-j)+1} + \sum_{j=k+1}^{2k+1} \frac{(-1)^j}{2(k-j)+1}\right| = 2\sum_{j=0}^{k} \frac{(-1)^j}{2j+1} \\
&\leq 2\left(\frac{\pi}{4} + \frac{1}{2k+3}\right) = \frac{\pi}{2} + \frac{2}{n+2},
\end{aligned}$$

and therefore

$$\Lambda_n\left(\frac{1}{2}\right) = \frac{N(\frac{1}{2})}{D(\frac{1}{2})} \geq \frac{2\ln(n+2)}{\pi + \frac{4}{n+2}}. \tag{8}$$

From (7) and (8) we finally conclude

$$\Lambda_n = \max_{0\leq x\leq 1} \Lambda_n(x) \geq \frac{2\ln(n+1)}{\pi + \frac{4}{n+1}} \geq \frac{2\ln(n+1)}{\pi + \frac{4}{n}} = \frac{2n}{4+n\pi}\ln(n+1)$$

for any $n \in \mathbb{N}$. □

Note that the bound in Theorem 1 is a considerable improvement of the corresponding result given by Berrut and Mittelmann [2, Theorem 3.1], namely $\Lambda_n \geq 1/(2n^2)$. Our next result concerns the upper bound of the Lebesgue constant.

Theorem 2. *The Lebesgue constant for interpolation with the basis functions $b_i(x)$ in (2) at the nodes x_j in (6) satisfies*

$$\Lambda_n \le 2 + \ln(n).$$

Proof. If $x = x_k$ for any k, then it follows from the interpolation property of the basis functions that $\Lambda_n(x) = 1$. So let $x_k < x < x_{k+1}$ for some k and consider the function

$$\Lambda_{n,k}(x) = \frac{(x-x_k)(x_{k+1}-x)\displaystyle\sum_{j=0}^{n}\frac{1}{|x-x_j|}}{\left|(x-x_k)(x_{k+1}-x)\displaystyle\sum_{j=0}^{n}\frac{(-1)^j}{x-x_j}\right|} =: \frac{N_k(x)}{D_k(x)}. \tag{9}$$

Our goal now is to bound the numerator $N_k(x)$ from above and the denominator $D_k(x)$ from below.

We first focus on the numerator,

$$\begin{aligned}
N_k(x) &= (x-x_k)(x_{k+1}-x)\sum_{j=0}^{n}\frac{1}{|x-x_j|} \\
&= (x-x_k)(x_{k+1}-x)\left(\sum_{j=0}^{k-1}\frac{1}{x-x_j} + \frac{1}{x-x_k} + \frac{1}{x_{k+1}-x} + \sum_{j=k+2}^{n}\frac{1}{x_j-x}\right) \\
&= (x_{k+1}-x) + (x-x_k) + (x-x_k)(x_{k+1}-x)\left(\sum_{j=0}^{k-1}\frac{1}{x-x_j} + \sum_{j=k+2}^{n}\frac{1}{x_j-x}\right) \\
&= (x_{k+1}-x_k) + (x-x_k)(x_{k+1}-x)\left(\sum_{j=0}^{k-1}\frac{1}{x-x_j} + \sum_{j=k+2}^{n}\frac{1}{x_j-x}\right).
\end{aligned}$$

As the nodes x_j are equally spaced with distance $h = 1/n$ we have

$$\frac{1}{x_i-x_j} = \frac{1}{h(i-j)} = \frac{n}{i-j}$$

for any $i \ne j$ and

$$(x-x_k)(x_{k+1}-x) \le \left(\frac{h}{2}\right)^2 = \frac{1}{4n^2}$$

for $x_k < x < x_{k+1}$. Therefore, using also (4), we get

$$\begin{aligned}
N_k(x) &\le \frac{1}{n} + \frac{1}{4n^2}\left(\sum_{j=0}^{k-1}\frac{1}{x_k-x_j} + \sum_{j=k+2}^{n}\frac{1}{x_j-x_{k+1}}\right) \\
&= \frac{1}{n} + \frac{1}{4n^2}\left(\sum_{j=0}^{k-1}\frac{n}{k-j} + \sum_{j=k+2}^{n}\frac{n}{j-k-1}\right) \\
&= \frac{1}{n} + \frac{1}{4n}\left(\sum_{j=1}^{k}\frac{1}{j} + \sum_{j=1}^{n-k-1}\frac{1}{j}\right) \\
&\le \frac{1}{n} + \frac{1}{4n}\big(\ln(2k+1) + \ln(2n-2k-1)\big) \\
&= \frac{1}{n} + \frac{1}{4n}\ln\big((2k+1)(2n-(2k+1))\big) \\
&\le \frac{1}{n} + \frac{1}{4n}\ln\big((2n/2)^2\big) \\
&= \frac{1}{n} + \frac{1}{2n}\ln(n).
\end{aligned}$$

We now turn to the denominator in (9), ignoring the absolute value and assuming both k and n to be even for the moment, so that

$$\begin{aligned}
D_k(x) &= (x-x_k)(x_{k+1}-x)\sum_{j=0}^{n}\frac{(-1)^j}{x-x_j} \\
&= (x-x_k)(x_{k+1}-x)\left(\sum_{j=0}^{k-1}\frac{(-1)^j}{x-x_j}+\frac{1}{x-x_k}+\frac{1}{x_{k+1}-x}-\sum_{j=k+2}^{n}\frac{(-1)^j}{x_j-x}\right) \\
&= \frac{1}{n}+(x-x_k)(x_{k+1}-x)\left(\sum_{j=0}^{k-1}\frac{(-1)^j}{x-x_j}-\sum_{j=k+2}^{n}\frac{(-1)^j}{x_j-x}\right).
\end{aligned}$$

Pairing the positive and negative terms in the rightmost factor adequately then gives

$$\begin{aligned}
S_k(x) &= \sum_{j=0}^{k-1}\frac{(-1)^j}{x-x_j}-\sum_{j=k+2}^{n}\frac{(-1)^j}{x_j-x} \\
&= \frac{1}{x-x_0}+\left(\frac{1}{x-x_2}-\frac{1}{x-x_1}\right)+\dots+\left(\frac{1}{x-x_{k-2}}-\frac{1}{x-x_{k-3}}\right)-\frac{1}{x-x_{k-1}} \\
&\quad -\frac{1}{x_{k+2}-x}+\left(\frac{1}{x_{k+3}-x}-\frac{1}{x_{k+4}-x}\right)+\dots+\left(\frac{1}{x_{n-1}-x}-\frac{1}{x_n-x}\right).
\end{aligned} \tag{10}$$

Since both the leading term and all paired terms are positive, we have

$$S_k(x) > -\frac{1}{x-x_{k-1}}-\frac{1}{x_{k+2}-x} \geq -\frac{1}{x_k-x_{k-1}}-\frac{1}{x_{k+2}-x_{k+1}} = -2n$$

and further

$$D_k(x) = \frac{1}{n}+(x-x_k)(x_{k+1}-x)S_k(x) \geq \frac{1}{n}+\frac{1}{4n^2}(-2n) = \frac{1}{n}-\frac{1}{2n} = \frac{1}{2n}.$$

This bound also holds if n is odd as this only adds a single positive term $1/(x_n-x)$ to $S_k(x)$ in (10), and if k is odd then a similar reasoning shows that $D_k(x) \leq -1/(2n)$. Therefore, we have $|D_k(x)| \geq 1/(2n)$ regardless of the parity of k and n, and combining the bounds for numerator and denominator in (9) yields

$$\Lambda_n = \max_{k=0,\dots,n-1}\left(\max_{x_k<x<x_{k+1}}\frac{N_k(x)}{D_k(x)}\right) \leq \frac{\frac{1}{n}+\frac{1}{2n}\ln(n)}{\frac{1}{2n}} = 2+\ln(n).$$

□

3 Numerical experiments

Besides the theoretical results in the previous section, we also performed a number of numerical experiments to further analyse the behaviour of the Lebesgue function and the Lebesgue constant of Berrut's rational interpolant.

Figure 1 shows the Lebesgue function for interpolation at $n+1$ equidistant nodes for some small values of n. The plots suggest that the maximum is always obtained near the centre of the interpolation interval, which explains why we analyse the Lebesgue function at $x=1/2$ for odd n and at $x=1/2+h/2$ for even n in the proof of Theorem 1.

We further computed the Lebesgue constant numerically for $1 \leq n \leq 200$ by evaluating

$$\Lambda_n \approx \max_{0\leq k\leq N}\Lambda_n\left(\frac{k}{N}\right)$$

for $N = 10\,000\,n$. Figure 2 shows these values as well as the lower and upper bound from Theorem 1 and Theorem 2. Our results suggest that the sequences $(\Lambda_{2k-1})_{k\in\mathbb{N}}$ and $(\Lambda_{2k})_{k\in\mathbb{N}}$ are strictly increasing and that $\Lambda_{2k} < \Lambda_{2k-1}$ for $k \geq 2$. Thus, interpolation at an odd number of equidistant nodes (i.e. n even) is slightly more stable than interpolation at an even number of nodes, which could be related to the fact that Berrut's rational

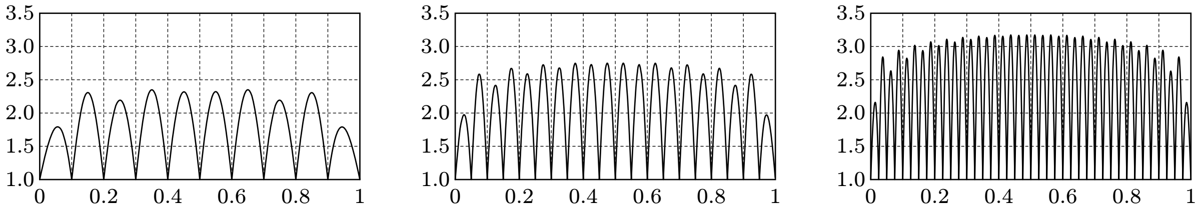

Figure 1: Lebesgue function of Berrut's interpolant at $n+1$ equidistant nodes for $n=10,20,40$.

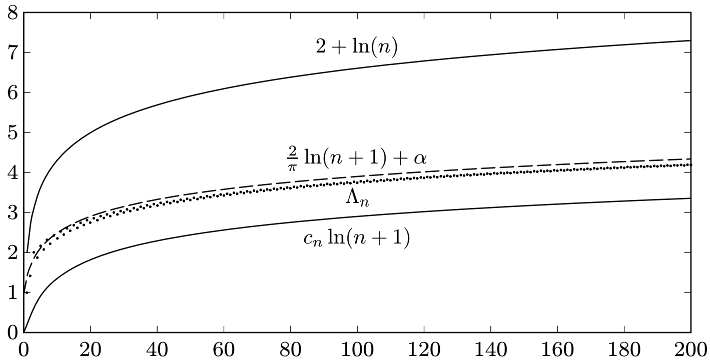

Figure 2: Lebesgue constant of Berrut's interpolant at $n+1$ equidistant nodes for $1 \le n \le 200$, compared to our lower and upper bound and Rivlin's lower bound for polynomial interpolation at Chebyshev nodes.

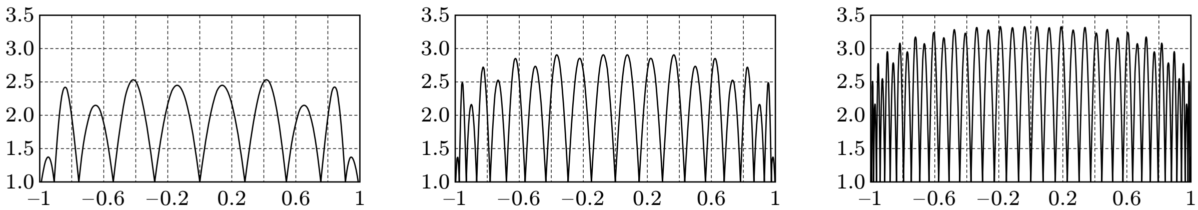

Figure 3: Lebesgue function of Berrut's interpolant at $n+1$ Chebyshev nodes for $n=10,20,40$.

interpolant reproduces only constant functions for even n, but linear functions for odd n (which follows from Theorem 3 in [6]). Moreover, we observe that for $n \ge 10$ the Lebesgue constant of this rational interpolant is even smaller than the lower bound for the Lebesgue constant for polynomial interpolation at Chebyshev nodes that was found by Rivlin [7], namely $\frac{2}{\pi}\ln(n+1)+\alpha$ with $\alpha \approx 0.9625$.

Interestingly, the Lebesgue constant of Berrut's rational interpolant behaves very similarly if we consider interpolation at the Chebyshev nodes

$$x_j = \cos\left(\frac{2j+1}{2n+2}\pi\right), \qquad j=0,\dots,n$$

and the logarithmically distributed nodes

$$x_j = \ln\left(1+\frac{i}{n}(e-1)\right), \qquad j=0,\dots,n.$$

The corresponding Lebesgue functions for some small values of n are plotted in Figure 3 and Figure 4, respectively. Figure 5 shows the numerically computed Lebesgue constants, together with the two functions $\frac{2}{\pi}\ln(n+1)+0.6$ and $\frac{2}{\pi}\ln(n+1)+1.2$ as a reference to help comparing the two plots. As for equidistant nodes, it seems that $(\Lambda_{2k-1})_{k\in\mathbb{N}}$ and $(\Lambda_{2k})_{k\in\mathbb{N}}$ are strictly increasing and that $\Lambda_{2k} < \Lambda_{2k-1}$ for $k \ge 2$ in both cases.

Overall, we observe that the Lebesgue constants for Chebyshev points are greater than those for logarithmically distributed points, which in turn are slightly greater than the Lebesgue constants for equidistant points. However, in all three cases, the asymptotic growth seems to be $\frac{2}{\pi}\ln(n+1)$. We will verify this conjecture and study the Lebesgue constant of Berrut's rational interpolant for general point distributions in a forthcoming paper.

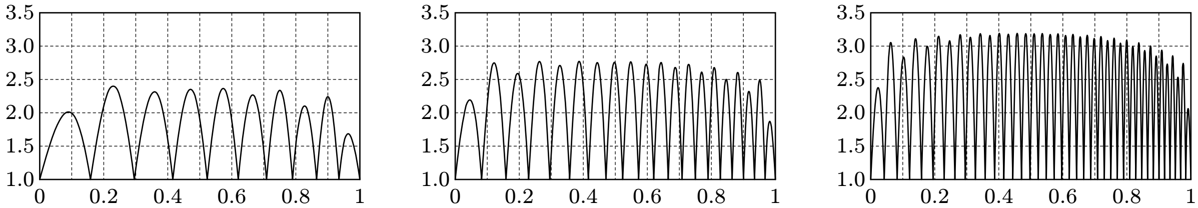

Figure 4: Lebesgue function of Berrut's interpolant at $n+1$ logarithmic nodes for $n = 10, 20, 40$.

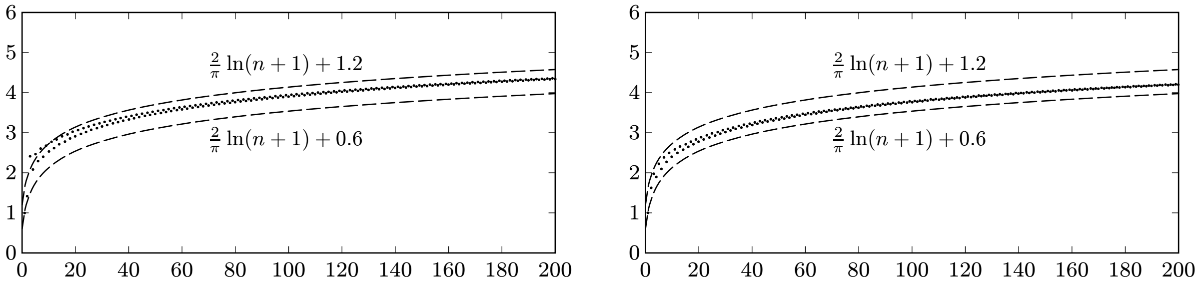

Figure 5: Lebesgue constant of Berrut's interpolant at $n+1$ Chebyshev (left) and logarithmic nodes (right) for $1 \le n \le 200$, compared to two logarithmic functions.

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References

[1] J.-P. Berrut. Rational functions for guaranteed and experimentally well-conditioned global interpolation. *Comput. Math. Appl.*, 15(1):1–16, 1988.

[2] J.-P. Berrut and H. D. Mittelmann. Lebesgue constant minimizing linear rational interpolation of continuous functions over the interval. *Comput. Math. Appl.*, 33(6):77–86, Mar. 1997.

[3] L. Brutman. On the Lebesgue functions for polynomial interpolation. *SIAM J. Numer. Anal.*, 15(4):694–704, Aug. 1978.

[4] L. Brutman. Lebesgue functions for polynomial interpolation – a survey. *Annals Numer. Math.*, 4:111–127, 1997.

[5] J. M. Carnicer. Weighted interpolation for equidistant points. *Numer. Algorithms*, 55(2–3):223–232, Nov. 2010.

[6] M. S. Floater and K. Hormann. Barycentric rational interpolation with no poles and high rates of approximation. *Numer. Math.*, 107(2):315–331, Aug. 2007.

[7] T. J. Rivlin. The Lebesgue constants for polynomial interpolation. In H. G. Garnir, K. R. Unni, and J. H. Williamson, editors, *Functional Analysis and its Applications*, volume 399 of *Lecture Notes in Mathematics*, pages 422–437. Springer, Berlin, 1974.

[8] A. Schönhage. Fehlerfortpflanzung bei Interpolation. *Numer. Math.*, 3:62–71, 1961.

[9] S. J. Smith. Lebesgue constants in polynomial interpolation. *Annales Math. Inf.*, 33:109–123, 2006.

[10] L. N. Trefethen and J. A. C. Weideman. Two results on polynomial interpolation in equally spaced points. *Journal of Approximation Theory*, 65(3):247–260, June 1991.

[11] A. H. Turetskii. The bounding of polynomials prescribed at equally distributed points. *Proc. Pedag. Inst. Vitebsk*, 3:117–127, 1940.

[12] Q. Wang, P. Moin, and G. Iaccarino. A rational interpolation scheme with superpolynomial rate of convergence. *SIAM J. Numer. Anal.*, 47(6):4073–4097, 2010.